Please check the examination details below before entering your candidate information

Candidate surname	Other names

Pearson Edexcel Level 3 GCE

Centre Number | Candidate Number

Monday 5 Oct 2020

Morning (Time: 1 hour 30 minutes) | Paper Reference **9FM0/01**

Further Mathematics

Advanced

Paper 1: Core Pure Mathematics 1

You must have:
Mathematical Formulae and Statistical Tables (Green), calculator

Total Marks

Candidates may use any calculator permitted by Pearson regulations. Calculators must not have the facility for algebraic manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided
  – there may be more space than you need.
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Inexact answers should be given to three significant figures unless otherwise stated.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 7 questions in this question paper. The total mark for this paper is 75.
- The marks for **each** question are shown in brackets
  – use this as a guide as to how much time to spend on each question.

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.

Turn over

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Pearson

1. $$f(z) = 3z^3 + pz^2 + 57z + q$$

where p and q are real constants.

Given that $3 - 2\sqrt{2}i$ is a root of the equation $f(z) = 0$

(a) show all the roots of $f(z) = 0$ on a single Argand diagram, **(7)**

(b) find the value of p and the value of q. **(3)**

(a) notice we are given a cubic equation, which according to the Fundamental Law of Algebra can have the following combination of roots:

- 3 real roots
- a complex conjugate pair and a real root

because we are already given a complex root - call it $z_1 = 3 - 2\sqrt{2}i$, we know we are already looking at the second option

↳ first finding the complex conjugate : $z^* = 3 + 2\sqrt{2}i$

now 2 ways to find the real root:

WAY 1: using roots of polynomials formulae

given cubic which we try to apply our roots of polynomials formulae to:

$$f(z) = 3z^3 + \overset{\Sigma\alpha}{p}z^2 + \overset{\Sigma\alpha\beta}{57}z + \overset{\alpha\beta\gamma}{q}$$

Sum of roots : $\Sigma\alpha = -b/a = -p/3$

sum of product pairs : $\Sigma\alpha\beta = c/a = 57/3 = 19$

product of roots : $\alpha\beta\gamma = -d/a = -q/3$

using what we know from above, i.e that

$\Sigma\alpha\beta = 19$ and letting real root $= \gamma$

$$\Rightarrow \underbrace{(3-2\sqrt{2}i)(3+2\sqrt{2}i)}_{\text{using shortcut: } zz^* = a^2+b^2} + (3-2\sqrt{2}i)\gamma + (3+2\sqrt{2}i)\gamma = 19$$

$$17 + (3-2\sqrt{2}i)\gamma + (3+2\sqrt{2}i)\gamma = 19$$

$$\Rightarrow (3-2\sqrt{2}i)\gamma + (3+2\sqrt{2}i)\gamma = 19 - 17 = 2$$

factorise γ out :

$$\gamma(3 - 2\sqrt{2}i + 3 + 2\sqrt{2}i) = 2$$

$$\gamma(6) = 2 \qquad \div 6 \quad \div 6$$

$$\gamma = \frac{2}{6} = \frac{1}{3}$$

OR WAY 2: by inspection

first form a quadratic out of the two complex roots – using general formula:

$z^2 - (\alpha+\beta) + \alpha\beta$, where $z + z^* = 2a_{z_1}$,

$zz^* = a_{z_1}^2 + b_{z_1}^2$

here $z + z^* = 2(3)$

$= 6$

$zz^* = (3)^2 + (2\sqrt{2})^2$

$= 9 + 8$

$= 17$

$\Rightarrow z^2 - 6z + 17$

now know we have to multiply this by some linear term to get the cubic

4 looking at the coefficient of $f(z)$ see '3' $\Rightarrow$ the linear term is also being multiplied by '3', with the constant term unknown, so 'a'

$$(z^2 - 6z + 17)(3z + a) \equiv 3z^3 + pz^2 + 57z + q$$

...need to compare 'z' terms as only this is known:

$-6a + 51 = 57$

$\Rightarrow 6a = -6 \quad \div 6 \quad \div 6$

$\Rightarrow a = -1$

$\Rightarrow$ the linear term becomes $(3z - 1)$

making this equal 0 gets us $3z - 1 = 0$

$\Rightarrow 3z = 1 \quad \div 3 \quad \div 3$

$\Rightarrow z = 1/3$

$\therefore$ the real root $= 1/3$

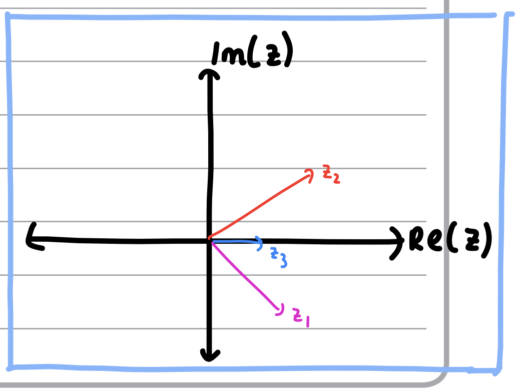
---

so know our 3 roots are $z_1 = 3 - 2\sqrt{2}i$

$z_2 = 3 + 2\sqrt{2}i$

$z_3 = 1/3$

plotting these on an Argand diagram:



(b) again 2 ways to use our three roots from part (a) to find 'p' and 'q'

WAY 1: roots of polynomials formulae

continuing from WAY 1 of part (a), can use:

· sum of roots $= \sum\alpha = -p/3$

and evaluating the LHS:

$$(3-2\sqrt{2}i)+(3+2\sqrt{2}i)+\frac{1}{3}$$

$$= 6+\frac{1}{3} = \frac{19}{3}$$

and equate to the RHS

$$\frac{19}{3} = -\frac{p}{3}$$

$$\Rightarrow -p = 19$$

$$\Rightarrow \boxed{p=-19}$$

now use:

· product of roots $= \alpha\beta\gamma = -q/3$

evaluate LHS

$$(3-2\sqrt{2}i)(3+2\sqrt{2}i)(\tfrac{1}{3})$$

$$= 17(\tfrac{1}{3}) = \frac{17}{3}$$

equate to RHS

$$\frac{17}{3} = -\frac{q}{3}$$

$$\Rightarrow -q = 17$$

$$\Rightarrow \boxed{q=-17}$$

WAY 2: by inspection

ctd. from WAY 2 part (a), we have:

$$(z^2-6z+17)(3z-1) = 3z^3+pz^2+57z+q$$

...for 'p':

...need to get z^2 term from LHS:

$$-1-18=-19$$

equate to RHS

$$\Rightarrow \boxed{p=-19}$$

Question 1 continued

... now for 'q' - need to compare constants:

$-17 = q$

$\boxed{\Rightarrow q = -17}$

WAY 3: using factor theorem if $(z-(3-2\sqrt{2}i))$ is a factor then $f(z)=0$

$f(3-2\sqrt{2}i) = 3(3-2\sqrt{2}i)^3 + p(3-2\sqrt{2}i)^2 + 57(3-2\sqrt{2}i) + q = 0$

① $3(3-2\sqrt{2}i)^3$

PASCAL'S

1 1

1 2 1

1 3 3 1

$3\left(1(3)^3 + 3(3^2 \times -2\sqrt{2}i) + 3(3 \times (-2\sqrt{2}i)^2) + 1(-2\sqrt{2}i)^3\right)$

$= 3(27 + (-54\sqrt{2}i) + (-72) + (16\sqrt{2}i))$

$= 3(-45 - 38\sqrt{2}i)$

$= -135 - 114\sqrt{2}i$

② $p(3-2\sqrt{2}i)^2$

$= p((3)^2 + 2(3 \times (-2\sqrt{2}i)) + (-2\sqrt{2}i)^2)$

$= p(9 - 12\sqrt{2}i - 8)$

$= p(1 - 12\sqrt{2}i)$

$= p - 12p\sqrt{2}i$

③ $57(3-2\sqrt{2}i)$

$= 171 - 144\sqrt{2}i$

④ q

... subbing into expansion:

$-135 - 114\sqrt{2}i + p - 12p\sqrt{2}i + 171 - 144\sqrt{2}i + q = 0$

collect real and imaginary terms:

$36 + p + q + i(-288\sqrt{2} - 12\sqrt{2}p) = 0$

make each real and imaginary part $= 0$

...real:

$36 + p + q = 0$

$\Rightarrow p + q = -36$ -①

...imaginary:

$-288\sqrt{2} - 12\sqrt{2}p = 0$

$\Rightarrow 12\sqrt{2}p = -288\sqrt{2}$

$\Rightarrow \boxed{p =} \frac{-288\cancel{\sqrt{2}}}{12\cancel{\sqrt{2}}} \boxed{= -19}$

sub into ①

$-19 + q = -36$

$\Rightarrow \boxed{q = -17}$

(Total for Question 1 is 10 marks)

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2. (a) Explain why $\int_1^{\infty} \frac{1}{x(2x+5)}\,dx$ is an improper integral. **(1)**

(b) Prove that

$$\int_1^{\infty} \frac{1}{x(2x+5)}\,dx = a\ln b$$

where a and b are rational numbers to be determined. **(6)**

(a) out of the 2 ways an integral is considered 'improper':

- ∞ in one of the limits
- when function that is to be integrated is undefined across the limit interval

we see that the first option is true

$\therefore$ integral improper due to the '∞' in the upper limit

(b) notice we are asked to integrate this improper integral... this must mean that we can get the integral to converge to a specific value - $a\ln b$

first let's work out the indefinite integration:

$$\int \frac{1}{x(2x+5)}\,dx$$

see this requires the integration of a fractional expression (explained more in detail on pg.9 - end of question):

Fractional expressions

4a. Can I split the numerator?
Is there a single term in the denominator?
4b. Can I do partial fractions?
Does the denominator factorise?
4c. Can I do algebraic division?
Is the fraction improper?

- can't split the numerator because of the 1 $\therefore$ only option is to do partial fractions (step 4)

$$\frac{1}{x(2x+5)} = \frac{A}{x} + \frac{B}{2x+5}$$

$$\Rightarrow 1 = A(2x+5) + Bx$$

WAY 1: compare coefficients

... x:

$0 = 2A + B$ - ①

... constants:

$1 = 5A$

$\div 5 \quad \div 5$

$\Rightarrow \boxed{A = 1/5}$

WAY 2: by substitution

... make each bracket equal 0:

$2x + 5 = 0$

$\Rightarrow 2x = -5$

$\div 2 \quad \div 2$

sub ② into ①

$0 = 2(1/5) + B$

$0 = \frac{2}{5} + B$

$\Rightarrow \boxed{B = -2/5}$

$x = -5/2$

$1 = B(-5/2)$

$\Rightarrow -\frac{5}{2}B = 1$

$\div -\frac{5}{2} \quad \div -\frac{5}{2}$

$\Rightarrow \boxed{B = -2/5}$

$x = 0$

$1 = 5A$

$\div 5 \quad \div 5$

$\boxed{A = 1/5}$

replace $\int \frac{1}{x(2x+5)}\,dx$ with $\int \frac{1}{5x} - \frac{2}{5(2x+5)}\,dx$

→ now 2 methods to integrate

METHOD 1: using general result $\int \frac{1}{x} = \ln(x) + c$

$\int \frac{1}{5x} - \frac{2}{5(2x+5)}\,dx$ ...factorise $\frac{1}{5}$ out:

$\frac{1}{5}\int \frac{1}{x} - \frac{2}{2x+5}\,dx$ ...splitting this up: $\frac{1}{5}\int \frac{1}{x}\,dx - \int \frac{2}{2x+5}\,dx$

standard result

by reverse chain rule

consider: $\ln(2x+5)$

differentiate: $\frac{1}{2x+5} \times 2 = \frac{2}{2x+5}$

scale - no need

$= \frac{1}{5}\left[\ln(x) - \ln(2x+5)\right]$

...applying log quotient rule:

$= \frac{1}{5}\ln\left(\frac{x}{2x+5}\right) + c$

now applying the LIMITS:

$\int_0^\infty \frac{1}{5x} - \frac{2}{5(2x+5)}\,dx$

$= \lim_{t\to\infty}\left[\frac{1}{5}\ln\left(\frac{x}{2x+5}\right)\right]_1^t$

$= \lim_{t\to\infty}\left\{\frac{1}{5}\ln\left(\frac{t}{2t+5}\right) - \frac{1}{5}\ln\left(\frac{1}{2(1)+5}\right)\right\} = \lim_{t\to\infty}\left[\frac{1}{5}\ln\left(\frac{t}{2t+5}\right) - \frac{1}{5}\ln\left(\frac{1}{7}\right)\right]$

Question 2 continued

as $t \to \infty$, using L'hospital rule :

$$\frac{1}{5}\ln\left(\frac{\frac{t}{t}}{\frac{2t+5}{t}}\right) - \frac{1}{5}\ln\left(\frac{1}{7}\right)$$

$$= \frac{1}{5}\ln\left(\frac{1}{2}\right) - \frac{1}{5}\ln\left(\frac{1}{7}\right)$$ ...using log quotient rule :

$$= \frac{1}{5}\ln\left(\frac{\frac{1}{2}}{\frac{1}{7}}\right)$$

$$\boxed{= \frac{1}{5}\ln\left(\frac{7}{2}\right)}$$

where $a = \frac{1}{5}$, $b = 7/2$

METHOD 2: using integration by completing the square

expanding the brackets in the denominator and completing the square

$$\frac{1}{2x^2+5x} \quad \text{factorise the 2 out} \quad = \frac{1}{2(x^2+\frac{5}{2}x)} \quad \text{complete the square} \quad = \frac{1}{2(x+\frac{5}{4})^2-\frac{25}{16}}$$

notice this is in the form $\frac{1}{x^2-a^2}$ from the formula booklet

where $x \to x+5/4$ and $a^2 = \frac{25}{16}$

$$a = \frac{5}{4}$$

sub into integration result (formula booklet)

$$\int \frac{1}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|$$

$$= \frac{1}{2} \times \frac{1}{2(5/4)}\ln\left|\frac{x+5/4-5/4}{x+5/4+5/4}\right|$$

$$= \frac{1}{5}\ln\left|\frac{x}{x+\frac{10}{4}}\right|$$

$$\times 4 \qquad \times 4$$

$$= \frac{1}{5}\ln\left|\frac{\cancel{4}^{2}x}{\cancel{4}^{2}x+\cancel{10}^{5}}\right|$$

$$= \frac{1}{5}\ln\left|\frac{2x}{2x+5}\right| + c$$

now applying the 'improper integral' limits

$$\int_0^\infty \frac{1}{x(2x+5)}\,dx = \lim_{t\to\infty}\left[\frac{1}{5}\ln\left(\frac{2x}{2x+5}\right)\right]_1^t$$

evaluate at limits

$$= \lim_{t\to\infty}\left\{\frac{1}{5}\ln\left(\frac{2t}{2t+5}\right) - \frac{1}{5}\ln\left(\frac{2}{7}\right)\right\}$$

as $t\to\infty$, use L'hospital rule:

$$\frac{1}{5}\ln\left(\frac{\frac{2t}{t}}{\frac{2t}{t}+\frac{5}{t}}\right) - \frac{1}{5}\ln\left(\frac{2}{7}\right)$$

$$= \frac{1}{5}\ln\left(\frac{2}{2}\right) - \frac{1}{5}\ln\left(\frac{2}{7}\right)$$

$$= \underbrace{\frac{1}{5}\ln(1)}_{\text{undefined}} - \frac{1}{5}\ln\left(\frac{2}{7}\right)$$

$$\Rightarrow \text{Ans} = -\frac{1}{5}\ln\left(\frac{2}{7}\right)$$

use log power rule

$$= \frac{1}{5}\ln\left(\frac{2}{7}\right)^{-1}$$

and index law

$$\boxed{= \frac{1}{5}\ln\left(\frac{7}{2}\right)}$$

Reminders:

Students find fractions tough as fractions can be so many types.

Check first (and throughout the question) if you can simplify by:

- using basic indices rules to simplify and expand brackets
  - $x^a \times x^b = x^{a+b}$
  - $\frac{x^a}{x^b} = x^{a-b}$
  - $\frac{3}{5x}$ *means* $\frac{3}{5}x^{-1}$.
  - $(\sqrt[b]{x})^a$ or $\sqrt[b]{x^a} = x^{\frac{a}{b}}$
- Factorising and maybe cancel first
- Is there a single term in denominator?

  split fractions using $\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$ or $(a+b)c^{-1}$

Then ask yourself:

1. **Is it an easy power type?** $\int x^n dx = \frac{x^{n+1}}{n+1}$
2. **Is it ln (natural logarithm)?** Form $\int \frac{f'(x)}{f(x)}dx$

   To recognize these, the power in the denominator is (almost always) 1. When you bring the denominator up to the numerator using negative power indices rule you get a power of -1. By adding one to the power and dividing it, you'll end up dividing by zero which you can't do

   $$\int \frac{f'(x)}{f(x)}dx = \ln f(x) + C$$

   Method: copy $\ln$(denominator). Remember ignore then differentiate to check you get what is inside the integral - correct with numbers only, not variables and only correct by multiplying or dividing. We can ignore the pink part since the derivative 'pops' out when we differentiate and we know when we differentiate our answer it must be what is inside the integral).
3. **Is it bring up and harder power type?** Bring the power up and becomes the form $\int f'(x)f(x)^n\,dx = \frac{f(x)^{n+1}}{n+1} + C$

   Recognisable by a power in the denominator other than

   $$\int \frac{4x}{(2x^2-1)^3} = \int 4x(2x^2-10)^{-3}\,dx \text{ etc}$$
4. **Is it Partial fractions!** Recognisable by products in the denominator.

   Form 1 $\frac{\dots}{(cx+d)(ex+f)} = \frac{A}{cx+d} + \frac{B}{ex+f}$

   Form 2 $\frac{\dots}{(dx+e)(fx+g)^2} = \frac{A}{dx+e} + \frac{B}{fx+g} + \frac{C}{(fx+g)^2}$

   (only advanced courses have this form)

   Form 3 $\frac{\dots}{(dx+e)(fx^2+g)} = \frac{A}{dx+e} + \frac{Bx+C}{fx^2+g}$
5. **Is it divide first?** Recognisable by **two or more terms** in the denominator and also where we have the matching highest powers in both numerator and denominator or a higher power in the numerator
6. **Rewriting/adapting fraction in a clever way (split up the numerator to get two fractions)**
7. **Is it inverse trig? (may need to complete the square first)** Either use the inverse trig results below or use a trig substitution

$$\int \frac{1}{\sqrt{a^2-(bx)^2}}\,dx = \frac{1}{b}\sin^{-1}\left(\frac{bx}{a}\right) + C$$

$$\int -\frac{1}{\sqrt{a^2-(bx)^2}}\,dx = \frac{1}{b}\cos^{-1}\left(\frac{bx}{a}\right) + C$$

$$\int \frac{1}{a^2+(bx)^2}\,dx = \frac{1}{ab}\tan^{-1}\left(\frac{bx}{a}\right) + C$$

(Total for Question 2 is 7 marks)

3.

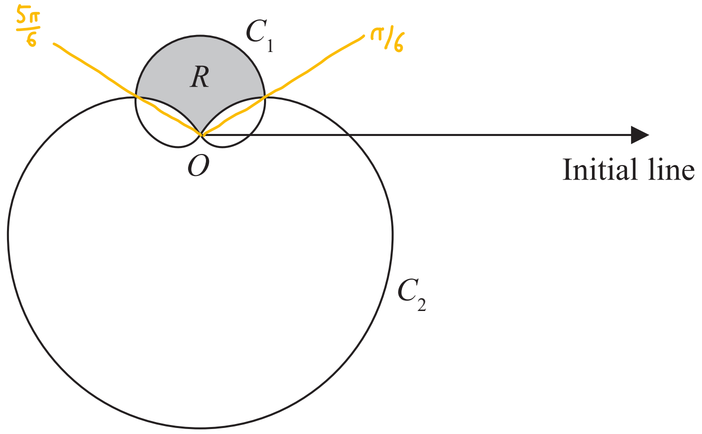


Figure 1

Figure 1 shows a sketch of two curves C_1 and C_2 with polar equations

$$C_1: r = (1 + \sin\theta) \qquad 0 \leqslant \theta < 2\pi$$

$$C_2: r = 3(1 - \sin\theta) \qquad 0 \leqslant \theta < 2\pi$$

The region R lies inside C_1 and outside C_2 and is shown shaded in Figure 1.

Show that the area of R is

$$p\sqrt{3} - q\pi$$

where p and q are integers to be determined.

(9)

we are asked to calculate the specific area between two polar curves - we know for definite that we will use the general formula for polar integration: $A = \pi\int_\alpha^\beta r^2\,d\theta$ - but for more information on the LIMITS we need to find the p.o.i between the two curves

$1+\sin\theta = 3(1-\sin\theta)$

expand the RHS

$1+\sin\theta = 3-3\sin\theta$

collect sines

$4\sin\theta = 2$

$\div 4 \qquad \div 4$

$\sin\theta = \frac{2}{4} = \frac{1}{2}$

$\Rightarrow \theta = \sin^{-1}\left(\frac{1}{2}\right)$

evaluate on calculator - only in the range

P 6 2 6 7 2 A 0 8 2 8

Question 3 continued

$\Rightarrow \theta = \pi/6$ or $(\pi - \pi/6) = 5\pi/6$

marking these on Fig 1

next consider the area radially - notice how

$$R = \int_{\pi/6}^{5\pi/6} C_1 \, d\theta - \int_{\pi/6}^{5\pi/6} C_2 \, d\theta$$

subbing into formula

$$R = \frac{1}{2}\int_{\pi/6}^{5\pi/6} (1+\sin\theta)^2 \, d\theta - \frac{1}{2}\int_{\pi/6}^{5\pi/6} \left(3(1-\sin\theta)\right)^2 d\theta$$

notice how because we are integrating over the same limits we can just rewrite above as one integral

$$= \frac{1}{2}\int_{\pi/6}^{5\pi/6} \left((1+\sin\theta)^2 - 9(1-\sin\theta)^2\right) d\theta$$

expand inside the brackets

$$= \frac{1}{2}\int_{\pi/6}^{5\pi/6} \left(1+2\sin\theta+\sin^2\theta - 9(1-2\sin\theta+\sin^2\theta)\right) d\theta$$

$$= \frac{1}{2}\int_{\pi/6}^{5\pi/6} (1+2\sin\theta+\sin^2\theta - 9 + 18\sin\theta - 9\sin^2\theta) \, d\theta$$

collect like terms

$$= \frac{1}{2}\int_{\pi/6}^{5\pi/6} (-8+20\sin\theta - 8\sin^2\theta) \, d\theta$$

↳ know we can't really integrate trig powers ∴ use cos double angle rearranged $\Rightarrow \sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$

$$= \frac{1}{2}\int_{\pi/6}^{5\pi/6} \left(-8+20\sin\theta - 8\left(\frac{1}{2} - \frac{1}{2}\cos 2\theta\right)\right) d\theta$$

$$= \frac{1}{2}\int_{\pi/6}^{5\pi/6} (-8+20\sin\theta - 4 + 4\cos 2\theta) \, d\theta$$

$$= \frac{1}{2}\int_{\pi/6}^{5\pi/6} (-12+20\sin\theta + 4\cos 2\theta) \, d\theta$$

multiply by a ½

$$= \int_{\pi/6}^{5\pi/6} (-6+10\sin\theta + 2\cos 2\theta) \, d\theta$$

integrate using $\int \cos k\theta = \frac{1}{k}\sin k\theta$ and $\int \sin k\theta \, d\theta = -\frac{1}{k}\cos k\theta$

Question 3 continued

$$= \left[-6\theta - 10\cos\theta + \sin 2\theta\right]_{\pi/6}^{5\pi/6}$$

$$= \left\{-6\left(\frac{5\pi}{6}\right) - 10\cos\left(\frac{5\pi}{6}\right) + \sin\left(\frac{5\pi}{3}\right)\right] - \left[-6\left(\pi/6\right) - 10\cos\left(\pi/6\right) + \sin\left(\pi/3\right)\right]\right\}$$

evaluate on calc

$$= -5\pi + \frac{9}{2}\sqrt{3} - \left(-\pi - \frac{9}{2}\sqrt{3}\right)\Big]$$

$$= -4\pi + 9\sqrt{3}$$

$$\boxed{\begin{array}{l} \text{or } 9\sqrt{3} - 4\pi \\ \Rightarrow p = 9,\ q = 4 \end{array}}$$

Question 3 continued

(Total for Question 3 is 9 marks)

4. The plane Π_1 has equation

$$\mathbf{r} = 2\mathbf{i} + 4\mathbf{j} - \mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) + \mu(-\mathbf{i} + 2\mathbf{j} + \mathbf{k})$$

where λ and μ are scalar parameters.

(a) Find a Cartesian equation for Π_1 **(4)**

The line l has equation

$$\frac{x-1}{5} = \frac{y-3}{-3} = \frac{z+2}{4}$$

(b) Find the coordinates of the point of intersection of l with Π_1 **(3)**

The plane Π_2 has equation

$$\mathbf{r}.(2\mathbf{i} - \mathbf{j} + 3\mathbf{k}) = 5$$

(c) Find, to the nearest degree, the acute angle between Π_1 and Π_2 **(2)**

(a) METHOD 1: Cartesian form general result

currently we are given the vector parametric form of the plane Π_1 - but the Cartesian form requires us to find the normal to the plane as well as the 'd' = a·n to finally give: $n_1x + n_2y + n_3z = d$

...two ways to find the normal to the plane:

WAY 1: vector method using fact that det(A) = 0

let normal $= \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

which, by definition, must be perpendicular to both the non-scalar vectors featured in the vector parametric form of the equation - hence the dot product of their components must equal 0

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} = x + 2y - 3z = 0$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = -x + 2y + z = 0$$

let $\boxed{z = 1}$ for both

WAY 2: vector cross product

to find the vector perpendicular to both non scalar vectors - just need to find the vector cross product of these two direction vectors

$$\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} \times \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = \begin{vmatrix} i & j & k \\ 1 & 2 & -3 \\ -1 & 2 & 1 \end{vmatrix}$$

$$= i\begin{vmatrix} 2 & -3 \\ 2 & 1 \end{vmatrix} - j\begin{vmatrix} 1 & -3 \\ -1 & 1 \end{vmatrix} + k\begin{vmatrix} 1 & 2 \\ -1 & 2 \end{vmatrix}$$

$$= \underline{i}(8) - \underline{j}(-2) + \underline{k}(4)$$

$$= 8\underline{i} + 2\underline{j} + 4\underline{k}$$

$\div 2$

$$= 4\underline{i} + \underline{j} + 2\underline{k} \text{ or } \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$$

Question 4 continued

$x + 2y = 3$ -①

$-x + 2y = -1$ -②

solving above by elimination

① - ② $\quad 2x = 4$

$\div 2 \quad \div 2$

$\boxed{x = 2}$

and subbing this 2 into any of ① or ②

$2 + 2y = 3$

$\Rightarrow 2y = 1$

$\div 2 \quad \div 2$

$\boxed{y = 1/2}$

$\Rightarrow$ normal $= \begin{pmatrix} 2 \\ 1/2 \\ 1 \end{pmatrix}$

multiply through by 2

$= \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$

now for 'd' need to take the position vector on the plane i.e $\begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix}$ and do $a \cdot n = d$

$\Rightarrow d = \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} = 2(4) + 4(1) + (-1)(2)$

$= 8 + 4 - 2$

$= 10$

$\Rightarrow$ general Cartesian formula for π_1:

$\boxed{4x + y + 2z = 10}$

METHOD 2: algebraically

getting the general coordinate for plane π_1

$r = \begin{pmatrix} 2 + \lambda - \mu \\ 4 + 2\lambda + 2\mu \\ -1 - 3\lambda + \mu \end{pmatrix}$

$\Rightarrow x = 2 + \lambda - \mu$ -①

$y = 4 + 2\lambda + 2\mu$ -②

$z = -1 - 3\lambda + \mu$ -③

and try to eliminate a parameter to finally get equations we can solve for $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

Question 4 continued

eliminating 'μ' from ① eliminating μ from ③

$\boxed{\mu = 2+\lambda - x}$ $\boxed{\mu = 1+3\lambda+z}$

and subbing both into ② to get two different expressions

$y = 4+2\lambda+2(2+\lambda-x)$

$\Rightarrow y = 4+2\lambda+4+2\lambda-2x$

$\Rightarrow y+2x = 8+4\lambda$ - ①

and $y = 4+2\lambda+2(1+3\lambda+z)$

$\Rightarrow y = 4+2\lambda+2+6\lambda+2z$

$\Rightarrow y-2z = 6+8\lambda$ - ②

now eliminate λ from both ① and ② - from ① :

$4\lambda = y+2x-8$

× 2

$8\lambda = 2y+4x-16$

and equate to ② rearranged

$8\lambda = y-2z-6$

$\Rightarrow 2y+4x-16 = y-2z-6$

finally collect like terms and integers on RHS

$\boxed{4x+y+2z=10}$

(b) METHOD 1 : using scalar product form of a plane

to find the point of intersection between line and plane, first need the scalar product form of π_1 ($\mathbf{r}\cdot\mathbf{n} = d$)

↳ so from (a)

$\pi_1 : \mathbf{r}\cdot\begin{pmatrix}4\\1\\2\end{pmatrix} = 10$

and then the general equation for the line from the given cartesian form - first need it in vector parametric form (the numerator negated becomes the position vector and the denominator is the direction vector)

$l : \mathbf{r} = \begin{pmatrix}1\\3\\-2\end{pmatrix} + \lambda\begin{pmatrix}5\\-3\\4\end{pmatrix}$

which as a general coordinate :

$$= \quad r = \begin{pmatrix} 1+5\lambda \\ 3-3\lambda \\ -2+4\lambda \end{pmatrix}$$

subbing this into scalar product form of the plane π_1

$$\begin{pmatrix} 1+5\lambda \\ 3-3\lambda \\ -2+4\lambda \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} = 10$$

evaluating the dot product

$$= 4(1+5\lambda) + (3-3\lambda) + 2(-2+4\lambda) = 10$$

expand the brackets

$$= 4 + 20\lambda + 3 - 3\lambda - 4 + 8\lambda = 10$$

$$\Rightarrow 25\lambda = 10 - 4 - 3 + 4$$

$$\Rightarrow 25\lambda = 7$$

$\div 25 \qquad \div 25$

$$\Rightarrow \boxed{\lambda = 7/25}$$

now we know value of the parameter at which the p.o.i occurs - subbing this into general coordinate on ℓ

$$\text{p.o.i} = \begin{pmatrix} 1+5(7/25) \\ 3-3(7/25) \\ -2+4(7/25) \end{pmatrix}$$

$$= \begin{pmatrix} 1+7/5 \\ 3-21/25 \\ -2+28/25 \end{pmatrix} = \boxed{\begin{pmatrix} 12/5 \\ 54/25 \\ -22/25 \end{pmatrix}}$$

METHOD 2: algebraically (using ℓ and π_1 Cartesian equations)

using given Cartesian equation for line ℓ to get expressions for 'y' and 'z' which we can sub into part (a)'s Cartesian equation and solve for 'x'

$$\frac{x-1}{5} = \frac{y-3}{-3} \quad -① \qquad\qquad \frac{x-1}{5} = \frac{z+2}{4}$$

cross multiply

$$-3(x-1) = 5(y-3)$$

expand

$$-3x + 3 = 5y - 15$$

$$\Rightarrow 5y = -3x + 18$$

factorise -3 out and $\div 5$ for 'y'

$$\Rightarrow \boxed{y = \frac{-3}{5}(x-6)}$$

cross multiply

$$4(x-1) = 5(z+2)$$

expand brackets

$$4x - 4 = 5z + 10$$

rearrange for 'z'

$$5z = 4x - 14$$

$\div 5 \qquad \div 5$

$$\boxed{z = \frac{1}{5}(4x-14)}$$

Sub into Cartesian equation of π_1

$$4x - \frac{3}{5}(x-6) + 2\left(\frac{1}{5}(4x-14)\right) = 10$$

expand brackets

$$4x - \frac{3}{5}x + \frac{18}{5} + \frac{8}{5}x - \frac{28}{5} = 10$$

Question 4 continued

collect 'x' terms

$$5x = 12$$

$\div 5 \quad \div 5$

$$\Rightarrow x = \frac{12}{5}$$

Subbing into the 'y' expression

$$y = -\frac{3}{5}\left(\frac{12}{5} - 6\right)$$

$$= -\frac{3}{5}\left(-\frac{18}{5}\right)$$

$$= \frac{54}{25}$$

and finally $z = \frac{1}{5}\left(4\left(\frac{12}{5}\right) - 14\right)$

$$= \frac{1}{5}\left(-\frac{22}{5}\right) = -\frac{22}{25}$$

$$\Rightarrow \boxed{\text{p.o.i} = \begin{pmatrix} 12/5 \\ 54/25 \\ -22/25 \end{pmatrix}}$$

(b) remembering the formula needed to calculate the acute angle between 2 planes: $\cos\theta = \dfrac{n_1 \cdot n_2}{|n_1||n_2|}$

we know from (a) that $n_1 = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$ and from given scalar product $n_2 = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$

Subbing this into the formula

$$\cos\theta = \frac{\begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}}{\sqrt{4^2 + (1)^2 + (2)^2}\sqrt{2^2 + (-1)^2 + (3)^2}}$$

$$\cos\theta = \frac{4(2) + 1(-1) + 2(3)}{\sqrt{21}\sqrt{14}}$$

$$\cos\theta = \frac{13}{\sqrt{21}\sqrt{14}}$$

$$\theta = \cos^{-1}\left(\frac{13}{\sqrt{21}\sqrt{14}}\right)$$

$$= 40.6963\ldots$$

$$\boxed{= 41^\circ \text{ (2 s.f)}}$$

(Total for Question 4 is 9 marks)

5. Two compounds, X and Y, are involved in a chemical reaction. The amounts in grams of these compounds, t minutes after the reaction starts, are x and y respectively and are modelled by the differential equations

$$\frac{dx}{dt} = -5x + 10y - 30 \quad - ①$$

$$\frac{dy}{dt} = -2x + 3y - 4 \quad - ②$$

(a) Show that

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = 50$$

(3)

(b) Find, according to the model, a general solution for the amount in grams of compound X present at time t minutes.

(6)

(c) Find, according to the model, a general solution for the amount in grams of compound Y present at time t minutes.

(3)

Given that $x = 2$ and $y = 5$ when $t = 0$

(d) find

(i) the particular solution for x,

(ii) the particular solution for y.

(4)

A scientist thinks that the chemical reaction will have stopped after 8 minutes.

(e) Explain whether this is supported by the model.

(1)

notice we are dealing with second order coupled differential equations where the 2ODE we are asked to prove is expressed in terms of 'x' ⇒ need to eliminate 'y'

∴ rearranging ① for 'y'

$$10y = \frac{dx}{dt} + 5x + 30$$

$\div 10$ $\qquad \div 10$

$$\Rightarrow y = \frac{1}{10}\frac{dx}{dt} + \frac{1}{2}x + 3$$

differentiate

$$\frac{dy}{dt} = \frac{1}{10}\frac{d^2x}{dt^2} + \frac{1}{2}\frac{dx}{dt}$$

and subbing into ②

Question 5 continued

$$\frac{1}{10}\frac{d^2x}{dt^2}+\frac{1}{2}\frac{dx}{dt}=-2x+3\left(\frac{1}{10}\frac{dx}{dt}+\frac{1}{2}x+3\right)-4$$

expand brackets

$$\frac{1}{10}\frac{d^2x}{dt^2}+\frac{1}{2}\frac{dx}{dt}=-2x+\frac{3}{10}\frac{dx}{dt}+\frac{3}{2}x+9-4$$

collect like terms (keep constants on RHS)

$$\frac{1}{10}\frac{d^2x}{dt^2}+\frac{1}{5}\frac{dx}{dt}+\frac{1}{2}x=5$$

x10 to get

$$\boxed{\frac{d^2x}{dt^2}+2\frac{dx}{dt}+5x=50}$$

(b) compound x - notice we have to solve part (a)'s non-homogenous 2ODE

$\boxed{\text{A.E}}$: $m^2+2m+5=0$

solve for 'm' – calc equtn solver or quadratic formula

$$m=\frac{-2\pm\sqrt{(2)^2-4(1)(5)}}{2(1)}$$

$$=\frac{-2\pm\sqrt{4-20}}{2}=\frac{-2\pm\sqrt{-16}}{2}$$

$$=\frac{-2\pm4i}{2}$$

$$\Rightarrow \boxed{m=-1\pm2i}$$

notice A.E solution consists of 2 complex numbers ∴ using the general formula for $m=\alpha\pm\beta i \Rightarrow x=e^{\alpha t}(A\cos\beta t+B\sin\beta t)$

$\boxed{\text{C.F}}$ $x=e^{-t}(A\cos2t+B\sin2t)$

...for P.I, looking at table:

let $x=\lambda$

$$\frac{dx}{dt}=0$$

Form of f(x)	Form of particular integral
k	λ
$ax+b$	$\lambda+\mu x$
ax^2+bx+c	$\lambda+\mu x+\nu x^2$
ke^{px}	λe^{px}
$m\cos\omega x$	$\lambda\cos\omega x+\mu\sin\omega x$
$m\sin\omega x$	$\lambda\cos\omega x+\mu\sin\omega x$
$m\cos\omega x+n\sin\omega x$	$\lambda\cos\omega x+\mu\sin\omega x$

WARNING! The particular integral must not contain any term in the complementary function. If it does, you'll need to add an x and possibly even an x² in front of your usual PI form

Question 5 continued $\frac{d^2x}{dt^2}=0$

sub into 2ODE

$0+2(0)+5\lambda=50$

$\Rightarrow 5\lambda=50$

$\div 5 \quad \div 5$

$\Rightarrow \lambda=10$

$\therefore$ G.S = C.F + P.I

$\Rightarrow x=e^{-t}(A\cos 2t+B\sin 2t)+10$

(c) for 'y' - can differentiate G.S from (b)

using the product rule - $\frac{d}{dt}(\sin kt)=k\cos kt$, $\frac{d}{dt}(\cos kt)=-k\sin kt$

$\frac{dx}{dt}=-e^{-t}(A\cos 2t+B\sin 2t)+e^{-t}(-2A\sin 2t+2B\cos 2t)$

and sub into rearranged ① for 'y' (used in (a))

$10y=\{e^{-t}(A\cos 2t+B\sin 2t)+e^{-t}(-2A\sin 2t+2B\cos 2t)\}+5$

$(e^{-t}(A\cos 2t+B\sin 2t)+10)+30$

$10y=e^{-t}(\cos 2t(-A+2B+5A)+\sin t(-B-2A+5B))+50+30$

$\div 10$

$\boxed{y=\frac{1}{10}e^{-t}((4A+2B)\cos 2t+\sin 2t(4B-2A))+8}$

(d) given initial conditions : when $t=0$, $x=2$

from (b)

$2=e^{-0}(A\cos(2\times 0)+B\sin(2\times 0))+10$

$\Rightarrow 2=A+10$

$\Rightarrow \boxed{A=-8}$

when $t=0$, $y=5$

from (c),

$5=\frac{1}{10}e^{-0}(\cos 2(0)(4(-8)+2B)+\sin(2\times 0)(4B-2A))+8$

$\Rightarrow 5=\frac{1}{10}(2B-32)+8$

$\times 10 \quad \times 10$

$50=2B-32+80$

$\Rightarrow 2B=2$

P 6 2 6 7 2 A 0 1 8 2 8

Question 5 continued

$\div 2 \quad \div 2$

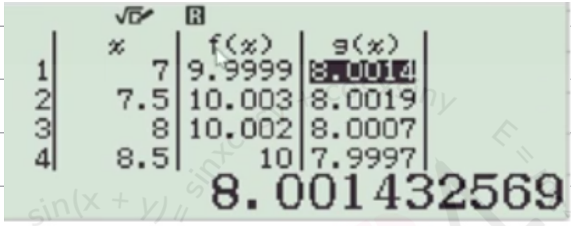
$\boxed{B = 1}$

$\therefore$ $\boxed{\begin{aligned} x &= e^{-t}(\sin 2t - 8\cos 2t) + 10 \\ y &= e^{-t}(2\sin 2t - 3\cos 2t) + 8 \end{aligned}}$

(e) if we let $f(x) = e^{-t}(\sin 2t - 8\cos 2t) + 10$

$g(x) = e^{-t}(2\sin 2t - 3\cos 2t) + 8$

and evaluate above at $t > 8$ on calc



	x	f(x)	g(x)
1	7	9.9999	8.0014
2	7.5	10.003	8.0019
3	8	10.002	8.0007
4	8.5	10	7.9997

8.001432569

x remains constant at 10

y remains constant at 8

$\boxed{\therefore \text{supports student's claim}}$

(Total for Question 5 is 17 marks)

6. (i) Prove by induction that for $n \in \mathbb{Z}^+$

$$\sum_{r=1}^{n}(3r+1)(r+2) = n(n+2)(n+3)$$

(6)

(ii) Prove by induction that for all positive **odd** integers n

$$f(n) = 4^n + 5^n + 6^n$$

is divisible by 15

(6)

proof by induction requires us to prove a conjecture is true for all $n \in \mathbb{N}$

(i) notice this is a summations proof

step 1: base step

prove true for $n=1$

LHS $\sum_{r=1}^{1}(3r+1)(r+2) = (3(1)+1)(1+2)$
$= (4)(3)$
$= 12$

RHS $1(1+2)(1+3)$
$= 1(3)(4)$
$= 12$

LHS = RHS $\therefore$ true for $n=1$

step 2: assumption step

assume true for $n=k$

$$\sum_{r=1}^{k}(3r+1)(r+2) = k(k+2)(k+3)$$

step 3: induction step

prove true for $n=k+1$

RHS: AIM:

$(k+1)(k+1+2)(k+1+3)$

$= (k+1)(k+3)(k+4)$

LHS: $\sum_{r=1}^{k+1}(3r+1)(r+2)$

splitting LHS to utilise $\sum_{r=1}^{k}$ from the assumption step

$= \sum_{r=1}^{k}(3r+1)(r+2) + (3(k+1)+1)(k+1+2)$

$= k(k+2)(k+3) + (3k+4)(k+3)$

factorise common factor

Question 6 continued

$= (k+3)[k(k+2)+(3k+4)]$

expand

$= (k+3)(k^2+5k+4)$

straight away see factoriseable

$= (k+3)(k+4)(k+1)$ = AIM (✓)

↳ same factors

$\therefore$ true for $n=k+1$

step 4: conclusion step

since true for $n=1$, if true for $n=k$ and true for $n=k+1$, then true for all $n \in \mathbb{N}$ //

(ii) now dealing with a divisibility proof - notice how instead of proving true for all $n \in \mathbb{N}$ we want to prove true for all +ve odd integers n

METHOD 1: using assumption case $f(2k+1)$

step 1: base case

prove true for $n=1$

$f(1) = 4^1 + 5^1 + 6^1$

$= 15(1)$ which is divisible by 15

$\therefore$ true for $n=1$

step 2: assumption case

assume true for $n=2k+1$ (i.e first odd integer)

$f(2k+1) = 4^{2k+1} + 5^{2k+1} + 6^{2k+1}$ which is divisible by 15 for all $k \in \mathbb{N}$

step 3: induction step

prove true for $n=2k+3$ (next odd integer)

WAY 1: power manipulation

$f(2k+3) = 4^{2k+3} + 5^{2k+3} + 6^{2k+3}$

splitting powers to get $f(2k+1)$ in

$= 4^{2k+1+2} + 5^{2k+1+2} + 6^{2k+1+2}$

evaluate indices

Question 6 continued

$= 4^{2k+1}(4^2) + 5^{2k+1}(5^2) + 6^{2k+1}(6^2)$

$= 16(4^{2k+1}) + 25(5^{2k+1}) + 36(6^{2k+1})$

now need to split coefficients up such that can factorise $f(2k+1)$ out and still be left with an expression that is a multiple of 15 $\therefore$ factorising 16 out:

$$16(4^{2k+1} + 5^{2k+1} + 6^{2k+1}) + \underbrace{9(5^{2k+1}) + 20(6^{2k+1})}$$

need to prove this is a multiple of 15 - splitting indices up

$$16f(2k+1) + 9(5^{2k} \cdot 5^1) + 20(6^{2k} \cdot 6^1)$$

$$= 16f(2k+1) + \underbrace{45(5^{2k}) + 120(6^{2k})}$$

multiples of 15!

factorising 15 out

$$\Rightarrow f(2k+3) = 16f(2k+1) + 15(3)(5^{2k+1}) + 15(8)(6^{2k+1})$$

which is divisible by 15

$\therefore$ true for $n = 2k+3$

WAY 2: more methodically : $f(2k+3) - f(2k+1)$

evaluating :

$f(2k+3) - f(2k+1) =$

$$4^{2k+3} + 5^{2k+3} + 6^{2k+3} - 4^{2k+1} - 5^{2k+1} - 6^{2k+1}$$

↳ splitting powers up to get assumed $f(2k+1)$ - then collecting like terms

$$= 4^{2k+1+2} + 5^{2k+1+2} + 6^{2k+1+2} - 4^{2k+1} - 5^{2k+1} - 6^{2k+1}$$

evaluate powers

$$= 4^2(4^{2k+1}) + 5^2(5^{2k+1}) + 6^2(6^{2k+1}) - 4^{2k+1} - 5^{2k+1} - 6^{2k+1}$$

$$= 16(4^{2k+1}) + 25(5^{2k+1}) + 36(6^{2k+1}) - 4^{2k+1} - 5^{2k+1} - 6^{2k+1}$$

collect like terms

$$= 15(4^{2k+1}) + 24(5^{2k+1}) + 35(6^{2k+1})$$

splitting indices (except first as see already multiple of 15)

$= 15(4^{2k+1}) + 24(5^1)(5^{2k}) + 35(6^1)(6^{2k})$

$= 15(4^{2k+1}) + 120(5^{2k}) + 210(6^{2k})$

$\Rightarrow f(2k+3) - f(2k+1) = 15(4^{2k+1}) + 120(5^{2k}) + 210(6^{2k})$

making $f(2k+3)$ the subject

$\Rightarrow f(2k+3) = f(2k+1) + 15(4^{2k+1}) + \underbrace{120(5^{2k}) + 210(6^{2k})}_{\text{need to prove these are multiples of 15}}$

$\Rightarrow f(2k+3) = f(2k+1) + 15(4^{2k+1}) + 15(8)(5^{2k}) + 15(14)(6^{2k})$

which is divisible by 15 $\therefore$ true for $n = 2k+3$

---

step 4: conclusion step

since true for $n=1$, if true for $n=k$ and true for $n=k+1$, then true for all $n \in \mathbb{N}$ //

METHOD 2: using assumption step $f(k)$

Step 1: base case

prove true for $n=1$

$f(1) = 4^1 + 5^1 + 6^1$

$= 15 = 15(1)$

$\therefore$ true for $n=1$

step 2: assumption step

realising that 'for all +ve odd numbers' suggests that 'n' is odd $\therefore$ we don't actually need to construct k as 2k+1 as 'n' is already assumed to be the first +ve odd integer

$\Rightarrow$ assume true for $n=k$

$f(k) = 4^k + 5^k + 6^k$

step 3: induction step - if $n=k$ is the first odd number $\Rightarrow n=k+2$ must be the next

INDUCTIVE case

$\therefore$ prove true for $n=k+2$

WAY 1: power manipulation to get $f(k)$

$f(k+2) = 4^{k+2} + 5^{k+2} + 6^{k+2}$

Splitting powers to get $f(k)$

$= 4^{k+2} + 5^{k+2} + 6^{k+2}$

splitting indices

$= (4^2)4^k + (5^2)5^k + (6^2)6^k$

$= 16(4^k) + 25(5^k) + 36(6^k)$ which after the same manipulation illustrated in METHOD 1, WAY 1, finally get:

Question 6 continued

$$16f(k) + 15\left(3(5^{2k}) + 8(6^{2k})\right)$$

$\therefore$ true for $n=k$

WAY 2: more methodically $f(k+2) - f(k)$

$f(k+2) - f(k)$

$= 4^{k+2} + 5^{k+2} + 6^{k+2} - 4^k - 5^k - 6^k$

splitting up the indices

$= 4^k(4^2) + 5^k(5^2) + 6^k(6^2) - 4^k - 5^k - 6^k$

$= 16(4^k) + 25(5^k) + 36(6^k) - 4^k - 5^k - 6^k$

collect like terms

$= 15(4^k) + 24(5^k) + 35(6^k)$

which after the same manipulation of coefficients as shown in WAY 2, method 1, finally gives:

$$f(k) + 15(4^{2k+1}) + 15\left(8(5^{2k}) + 14(6^{2k})\right)$$

$\therefore$ true for $n=k+1$

Step 4: conclusion step

since true for $n=1$, if true for $n=k$ and true for $n=k+1$, then true for all $n \in$ +ve odd numbers

(Total for Question 6 is 12 marks)

7. A sample of bacteria in a sealed container is being studied.

The number of bacteria, P, in thousands, is modelled by the differential equation

$$(1+t)\frac{dP}{dt} + P = t^{\frac{1}{2}}(1+t)$$

where t is the time in hours after the start of the study.

Initially, there are exactly 5000 bacteria in the container.

$\Rightarrow t = 5$

(a) Determine, according to the model, the number of bacteria in the container 8 hours after the start of the study.

(6)

(b) Find, according to the model, the rate of change of the number of bacteria in the container 4 hours after the start of the study.

(4)

(c) State a limitation of the model.

(1)

(a) notice we are given a 1ODE - can't separate the variables as involves an addition rather than the product of P and its derivatives

next check for reverse product rule on LHS

$\frac{d}{dt}(P) = \frac{dP}{dt}$ (v)

$$(1+t)\frac{dP}{dt} + P = t^{1/2}(1+t)$$

$\frac{d}{dt}(1+t) = 1$ (v)

$\therefore$ YES, CAN rewrite using $\frac{d}{dt}(P(1+t))$

NOTE: if hadn't spotted this - could've $\div(1+t)$ to get 1ODE in form $\frac{dy}{dx} + Py = Q$ and multiplied through by I.F - although this is mentioned in the main part of the MS this is way more time consuming than just noticing the reverse product rule straight away

$$\int \frac{d}{dt}(P(1+t))\,dt = \int t^{1/2}(1+t)\,dt$$

expanding RHS integral

$$\int \frac{d}{dt}(P(1+t))\,dt = \int t^{1/2} + t^{3/2}\,dt$$

integrating both sides

$\Rightarrow \boxed{G.S:}\ P(1+t) = \frac{2}{3}t^{3/2} + \frac{2}{5}t^{5/2} + c$

Question 7 continued

now subbing in the initial conditions - remember fact that the units are thousands

when $t=0$, $P=5$

$$5(1+0) = \frac{2}{3}(0)^{3/2} + \frac{2}{5}(0)^{5/2} + c$$

$$\Rightarrow 5 = c$$

$$\Rightarrow P(1+t) = \frac{2}{3}t^{3/2} + \frac{2}{5}t^{5/2} + 5$$

rearrange to make 'P' the subject

$\div 1+t$ $\quad\div 1+t$

$$\boxed{P.S:}\ P = \frac{\frac{2}{3}t^{3/2} + \frac{2}{5}t^{5/2} + 5}{1+t}$$

but the question is asking for P when $t=8$, subbing this into above P.S

$$P = \frac{\frac{2}{3}(8)^{3/2} + \frac{2}{5}(8)^{5/2} + 5}{1+8}$$

$= 10.27643\ldots$ thousand

i.e $10,276.43\ldots$

$\boxed{= 10,277 \text{ bacteria}}$

(b) 'rate of change' requires us to differentiate our P.S from part (a)

WAY 1: quotient rule

$$P = \frac{\frac{2}{3}t^{3/2} + \frac{2}{5}t^{5/2} + 5}{(1+t)}$$

$u = \frac{2}{3}t^{3/2} + \frac{2}{5}t^{5/2} + 5$ $\qquad v = 1+t$

$u' = t^{1/2} + t^{3/2}$ $\qquad v' = 1$

following quotient rule:

$$\frac{d}{dt}\left(\frac{u}{v}\right) = \frac{vu' + v'u}{v^2}$$

$$\frac{dP}{dt} = \frac{(1+t)(t^{1/2} + t^{3/2}) - \left(\frac{2}{3}t^{3/2} + \frac{2}{5}t^{5/2} + 5\right)}{(1+t)^2}$$

Substituting $t=4$ into above

$$\frac{dP}{dt} = \frac{(1+4)(4^{1/2}+4^{3/2}) - (\frac{2}{3}(4)^{3/2} + \frac{2}{5}(4)^{5/2} + 5)}{(1+4)^2}$$

evaluate on calc

$$= \frac{(5)(10) - (\frac{16}{3} + \frac{64}{5} + 5)}{25}$$

$$= \frac{\frac{403}{15}}{25} = \frac{403}{375}$$

but need thousands, so $\times 1000$

$$= \frac{3224}{3} = 1074.666..$$

$$\boxed{= 1070 \text{ (3s.f) bacteria per hr}}$$

WAY 2: using 1ODE

we can get rate of change by REARRANGING the 1ODE for $\frac{dP}{dt}$

know from part (a)'s P.S that $P = \dfrac{\frac{2}{3}t^{3/2} + \frac{2}{5}t^{5/2} + 5}{(1+t)}$

which at $t=4$ is

$$P = \frac{\frac{2}{3}(4)^{3/2} + \frac{2}{5}(4)^{5/2} + 5}{(1+4)}$$

$$= \frac{\frac{16}{3} + \frac{64}{5} + 5}{5} = \frac{\frac{347}{15}}{5} = \boxed{\frac{347}{75}}$$

subbing into 1ODE with $t=0$

$$(1+4)\frac{dP}{dt} + \frac{347}{75} = (4)^{1/2}(1+4)$$

Question 7 continued

$$\Rightarrow 5\frac{dP}{dt} + \frac{347}{75} = 2(5)$$

$$\Rightarrow 5\frac{dP}{dt} = 10 - \frac{347}{75}$$

$$\Rightarrow 5\frac{dP}{dt} = \frac{403}{75}$$

$$\div 5 \qquad \div 5$$

$$\frac{dP}{dt} = \frac{403}{375}$$

$$\frac{403}{375} \times 1000 = \frac{3224}{3} = 1{,}074.66\ldots$$

$$\boxed{= 1{,}075 \text{ bacteria/hr}}$$

(c) no. of bacteria increases indefinitely – not realistic //

P 6 2 6 7 2 A 0 2 7 2 8

Question 7 continued

(Total for Question 7 is 11 marks)

TOTAL FOR PAPER IS 75 MARKS